Computing project Part II

Assignment 3

We can use the same definitions for $\dot{q}$ and $\dot{p}$ from the previous assignments. The Hamiltonian is different now of course.

```
In[55]:= H[p_, q_, t_] := p^2/(2 m) + m g l (1 - Cos[q])
         q̇[p_, q_, eps_] := (H[p + eps, q, t] - H[p, q, t])/eps
         ṗ[p_, q_, eps_] := -(H[p, q + eps, t] - H[p, q, t])/eps
```

Define constants (m, l, g) and initial conditions:

```
In[58]:= m := 1;
         l := 1;
         g := 9.8;
         imax := 1000;
         t0 := 0;
         q0 := 1;
         p0 := 0;
         ϵ := .001;
         dt := .01;

In[67]:= T = Table[t0, {i, 1, imax}];
         Q = Table[q0, {i, 1, imax}];
         P = Table[p0, {i, 1, imax}];
```

■ 1

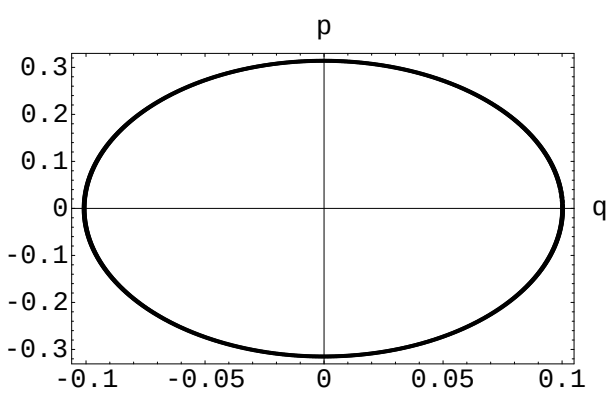
```
In[70]:= Q[[1]] = 0.1; P[[1]] = 0;
```

Make sure to use the predictor−corrector method, and make sure θ=q takes on values between $-\pi$ and π.

```
In[71]:= Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
          P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
          Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
          , {i, 2, imax}]

In[72]:= EE = Table[H[P[[i]], Q[[i]], T[[i]]], {i, 1, imax}];
```

Phase space plot:

```
In[73]:= ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```

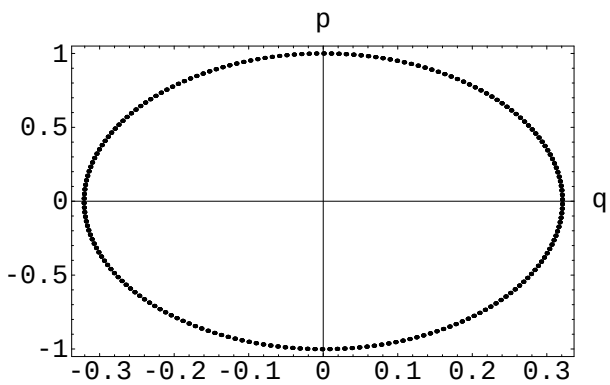


That looks very much like an harmonic oscillator, it’s pretty much an ellipse.

■ 2

Now with different initial conditions (the rest is just copy−paste from above).

```
In[74]:= Q[[1]] = 0; P[[1]] = 1;

In[75]:= Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
          P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
          Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
          , {i, 2, imax}]

In[76]:= ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```

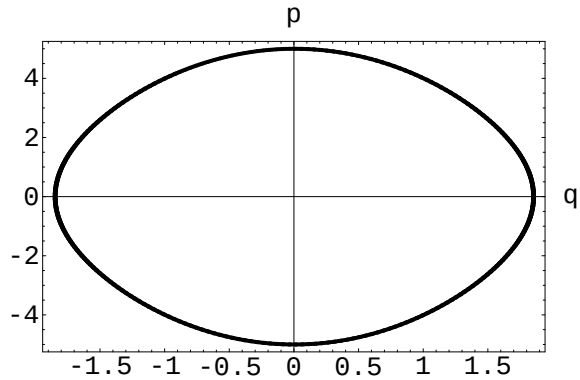


Still ellipse/harmonic oscillating behavior.

■ 3

Increase initial momentum:

```
In[77]:= Q[[1]] = 0; P[[1]] = 5;

In[78]:= Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
          P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
          Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
          , {i, 2, imax}]

In[79]:= ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```

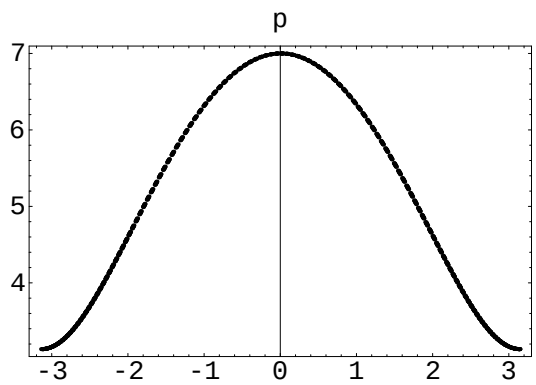


This does not look like a (perfect) ellipse anymore, but it is still oscillating behavior.

■ 4

Increase initial momentum even further:

```
In[80]:= Q[[1]] = 0; P[[1]] = 7;

In[81]:= Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
          P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
          Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
          , {i, 2, imax}]

In[82]:= ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```

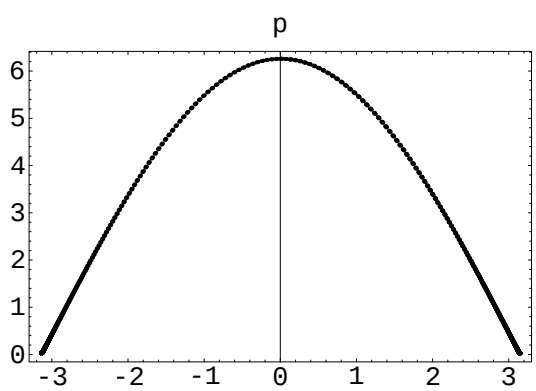


This is the second type of motion: the mass is going round in circles in real space. Momentum never becomes negative.

■ 4a

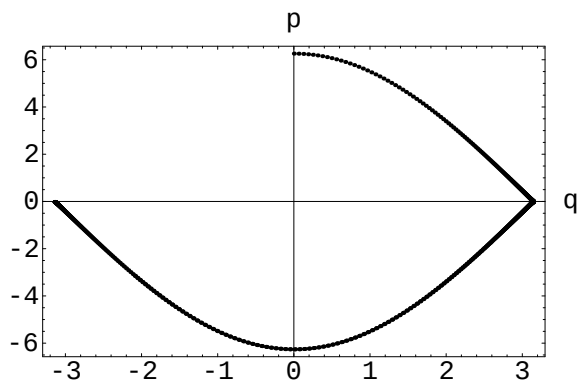
Now let’s try that initial momentum value of 6.26, which probably corresponds to the limiting trajectory.

```
In[83]:= Q[[1]] = 0; P[[1]] = 6.26;

In[84]:= Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
          Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
          P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
          Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
          , {i, 2, imax}]

In[85]:= ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```



This is roughly the limiting trajectory allright. Note that the momentum becomes (approximately) zero at q=θ=+/−π. Let's (analytically) calculate the corresponding initial momentum, with $U_{\max} = 2\,\mathrm{mg}$, $E = p^2/2\,m$, $E = U_{\max}$:

```
In[86]:=  p == Sqrt[4 m^2 g]

Out[86]=  p == 6.26099
```

Due to numerical approximations we made (discrete time steps, finite ϵ in calculation of time derivatives) our transition occurs at a value of p just below the exact value. The following graph is even closer to the limiting trajectory; both types of motion seem to occur, which is unphysical, of course, and demonstartes once more that we are making (small) approximations in our numerical calculation.

```
In[87]:=  Q[[1]] = 0; P[[1]] = 6.25995;

In[88]:=  Do[ T[[i]] = T[[i - 1]] + dt; predQ = Q[[i - 1]] + q̇[P[[i - 1]], Q[[i - 1]], ϵ] dt;
           predP = P[[i - 1]] + ṗ[P[[i - 1]], Q[[i - 1]], ϵ] dt;
           Q[[i]] = Q[[i - 1]] + (q̇[P[[i - 1]], Q[[i - 1]], ϵ] + q̇[predP, predQ, ϵ]) / 2 dt;
           P[[i]] = P[[i - 1]] + (ṗ[P[[i - 1]], Q[[i - 1]], ϵ] + ṗ[predP, predQ, ϵ]) / 2 dt;
           Q[[i]] = Mod[Q[[i]], 2 Pi, -Pi];
           , {i, 2, imax}]

In[89]:=  ListPlot[Transpose[{Q, P}], Frame → True, AxesLabel → {"q", "p"}];
```